

## NOTES

Computation of  $K_p(x)$ 

The modified Bessel function of the third kind,  $K_p(x)$ , is encountered in many physical problems. It can be computed by power series for small argument, and by asymptotic expansions for large argument [1]. However, it has been noted [2] that these methods do not yield reasonable accuracy for medium-sized real argument,  $x$ . It is particularly difficult to obtain any accuracy, even for a small  $x$ , when  $p$  is not an integer, but can approach integral values. It has, therefore, been suggested [2] to compute this function by numerical integration, but the integral used for this purpose was not a very convenient one, especially because of the additive form in which  $x$  appears in the integrand. A more convenient integration formula is introduced here.

Starting from the well-known [3] relation

$$K_p(z) = \frac{1}{2} \left(\frac{1}{2} z\right)^p \int_0^\infty \exp[-t - (z^2/4t)] t^{-p-1} dt, \quad (1)$$

changing the variable to  $2t/z$ , and taking the reciprocal in the region of integration 1 to  $\infty$ , one obtains

$$K_p(x) = \frac{1}{2} \int_0^1 t^{-1}(t^p + t^{-p}) \exp\left[-\frac{1}{2}x(t + t^{-1})\right] dt. \quad (2)$$

Numerical integration from Eq. (2) can be carried out very conveniently by the Gaussian quadrature formula [4], except for very large or very small values of  $x$ , for which computations by this method become more time-consuming than by other methods.

In a FORTRAN subroutine, written for the GOLEM—the electronic computer of this Institute, the range of integration, 0 to 1 is first divided into  $N = 8$  intervals, and the Gaussian quadrature formula is applied to each one of them. Then  $N$  is doubled, and integration is repeated. The iteration process terminates when the difference between an integration result and the previous one is less than this result times some predetermined parameter, specifying the required relative accuracy of the computations. However, if the required accuracy is not reached within  $NI = 10$  iterations, the process is terminated anyway, with a printout about the accuracy to which integration was carried out. The values specified here for  $N$  and  $NI$  are, of course rather arbitrary, and were chosen for a certain range of  $p$  and  $x$  used in a

particular problem. They can be changed if the required accuracy is not achieved. The subroutine was checked by comparing to various tabulated  $[I]$  values for integral  $p$ , and to values computed from equivalent analytical expression for half-integral index  $p$ . For all the values of  $x$  and  $p$  tried, the program converged to the correct answer, within the required tolerance of error, although in principle the method might [5] lead to the wrong answer in some cases.

RECEIVED: November 12, 1968

#### REFERENCES

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AMIKAM AHARONI

*Department of Electronics  
Weizmann Institute of Science  
Rehovot, Israel*